Generalized Connectivity of Star Graphs*

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Abstract

This paper shows that, for any integers n and k with $0 \leqslant k \leqslant n-2$, at least $(k+1)!(n-k-1)$ vertices or edges have to be removed from an n-dimensional star graph to make it disconnected and no vertices of degree less than k. The result gives an affirmative answer to the conjecture proposed by Wan and Zhang [Applied Mathematics Letters, 22 (2009), 264-267].

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1 Introduction

It is well known that interconnection networks play an important role in parallel computing/communication systems. An interconnection network can be modeled by a graph $G=(V,E)$, where V is the set of processors and E is the set of communication links in the network.

A subset $S \subset V(G)$ (resp. $F \subset E(G)$) of a connected graph G is called a *vertex-cut* (resp. *edge-cut*) if $G-S$ (resp. $G-F$) is disconnected. The *connectivity* $\kappa(G)$ (resp. *edge-connectivity* $\lambda(G)$) of G is defined as the minimum cardinality over all vertex-cuts (resp. edge-cuts) of G. The connectivity $\kappa(G)$ and edge-connectivity $\lambda(G)$ of a graph G are two important measurements for fault tolerance of the network since the larger $\kappa(G)$ or $\lambda(G)$ is, the more reliable the network is. Esfahanian [4] proposed the concept of restricted connectivity, Latifi *et al.* [6] generalized it to restricted k-connectivity which can measure fault tolerance of an interconnection network more accurately than the classical connectivity. The concepts stated here are slightly different from theirs.

A subset $S \subset V(G)$ (resp. $F \subset E(G)$) of a connected graph G, if any, is called a *k-vertex-cut* (resp. *edge-cut*), if $G-S$ (resp. $G-F$) is disconnected and has the minimum degree at least k. The *k-super connectivity* (resp. *edge-connectivity*) of G, denoted by

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$\kappa_s^{(k)}(G)$ (resp. $\lambda_s^{(k)}(G)$), is defined as the minimum cardinality over all k-vertex-cuts (resp. k-edge-cuts) of G. For any graph G and any integer k, determining $\kappa_s^{(k)}(G)$ and $\lambda_s^{(k)}(G)$ is quite difficult, there is no known polynomial algorithm to compute them yet. In fact, the existence of $\kappa_s^{(k)}(G)$ and $\lambda_s^{(k)}(G)$ is an open problem so far when $k \geqslant 1$. Only a little knowledge of results have been known on $\kappa_s^{(k)}$ and $\lambda_s^{(k)}$ for some special classes of graphs for any k.

As an attractive alternative network to the hypercube, the n-dimensional star graph S_n is proposed by Akers *et al.* [1]. Since it has superior degree and diameter to the hypercube as well as it is highly hierarchical and symmetrical [3], the star graph S_n has received considerable attention in recent years. In particular, Cheng and Lipman [2], Hu and Yang [5] and Rouskov *et al.* [8], independently, determined $\kappa_s^{(1)}(S_n) = 2n-4$ for $n \geqslant 3$. Yang *et al.* [11] proved $\lambda_s^{(2)}(S_n) = 6(n-3)$ for $n \geqslant 4$. Wan and Zhang [10] showed that $\kappa_s^{(2)}(S_n) = 6(n-3)$ for $n \geqslant 4$ and conjectured that $\kappa_s^{(k)}(S_n) = (k+1)!(n-k-1)$ for $k \leqslant n-2$. In this paper, we give an affirmative answer to the conjecture and generalize the above-mentioned results by proving that $\kappa_s^{(k)}(S_n) = \lambda_s^{(k)}(S_n) = (k+1)!(n-k-1)$ for any k with $0 \leqslant k \leqslant n-2$.

In Section 2, we recall the two structures of S_n and some lemmas to be used in our proofs. The proof of the main results is in Section 3. We conclude our work in Section 4.

2 Definitions and lemmas

For a given integer n with $n \geqslant 2$, let $I_n = \{1, 2, \ldots, n\}$, $I'_n = \{2, \ldots, n\}$ and $P(n) = \{p_1p_2\ldots p_n :\ p_i \in I_n, p_i \neq p_j, 1 \leqslant i \neq j \leqslant n\}$, the set of permutations on I_n. Clearly, $|P(n)| = n!$. For an element $p = p_1 \ldots p_j \ldots p_n \in P(n)$, the digit p_j is called the symbol in the j-th position (or dimension) in p.

The n-dimensional star graph, denoted by S_n, is an undirected graph with vertex-set $P(n)$. There is an edge between any two vertices if and only if their labels differ only in the first and another position. In other words, two vertices $u = p_1p_2\ldots p_i \ldots p_n$ and $v = p'_1p'_2\ldots p'_i\ldots p'_n$ are adjacent if and only if $v = p_ip_2\ldots p_{i-1}p_1p_{i+1}\ldots p_n$ for some $i \in I'_n$.

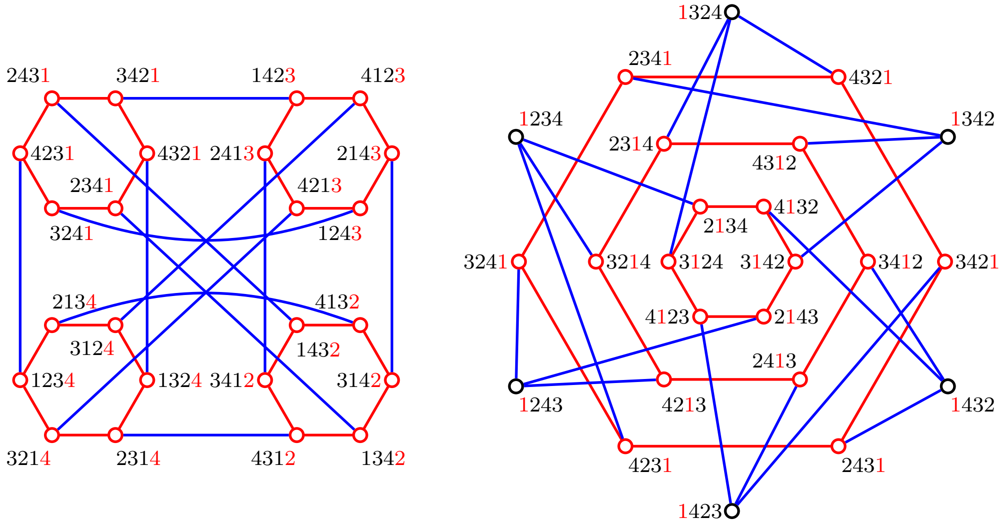
Like the hypercube, the star graph is a vertex- and edge-transitive graph with degree $(n-1)$ [**?**]. The following properties of S_n are very useful for our proof.

Lemma 2.1 (see Akers *et al.* [1],1989) $\kappa(S_n) = \lambda(S_n) = n-1$ *for* $n \geqslant 2$.

For a fixed symbol $i \in I_n$, let $S_n^{j:i}$ denote a subgraph of S_n induced by all vertices with symbol i in the j-th position for each $j \in I_n$. By the definition of S_n, it is easy to see that $S_n^{j:i}$ is isomorphic to S_{n-1} for each $j \in I'_n$ and $S_n^{1:i}$ is an empty graph with $(n-1)!$ vertices.

We will use two different hierarchical structures of S_n depending on different partition methods. The first one is partitioning along a fixed dimension, which is clear and used frequently. The second one is partitioning along a fixed symbol in I_n, which is a new structure proposed recently by Shi *et al.* [7].

Lemma 2.2 (The first structure) *For a fixed dimension* $j \in I'_n$, S_n *can be partitioned into* n *subgraphs* $S_n^{j:i}$, *which is isomorphic to* S_{n-1} *for each* $i \in I_n$. *Moreover, there are* $(n-2)!$ *independent edges between* $S_n^{j:i_1}$ *and* $S_n^{j:i_2}$ *for any* $i_1, i_2 \in I_n$ *with* $i_1 \neq i_2$.



Partitioned along dimension 4 *Partitioned along symbol 1*

Figure 1: Two structures of the 4-dimensional star graph S_4

Lemma 2.3 (Shi *et al.* [7], 2012, The second structure) *For a fixed symbol $i \in I_n$, S_n can be partitioned into n subgraphs $S_n^{j:i}$, which is isomorphic to S_{n-1} for each $j \in I'_n$ and $S_n^{1:i}$ is an empty graph with $(n-1)!$ vertices. Moreover, there are a perfect matching between $S_n^{1:i}$ and $S_n^{j:i}$ for any $j \in I'_n$, and there are no edge between $S_n^{j_1:i}$ and $S_n^{j_2:i}$ for any $j_1, j_2 \in I'_n$ with $j_1 \neq j_2$.*

It is easy to know that S_1, S_2, S_3 are isomorphic to K_1, K_2, C_6, respectively. S_4 is illustrated in Figure 1 by two different partition methods. As we will see, S_4 is partitioned along dimension 4 in the left figure, and is partitioned along symbol 1 in the right one.

3 Main results

In this section, we present our main results, that is, we determine the k-super connectivity and k-super edge connectivity of the n-dimensional star graph S_n. We first investigate the properties of subgraphs in S_n with minimum degree at least k. For a subset $X \subseteq V(S_n)$ and $j \in I_n$, we use U_j^X to denote the set of symbols in the j-th position of vertices in X, formally, $U_j^X = \{p_j : \ p_1 \dots p_j \dots p_n \in X\}$. The following lemma plays a key role in our proof.

Lemma 3.1 *Let H be a subgraph of S_n with vertex-set X and $k \in I_{n-1}$ a fixed integer. If $\delta(H) \geqslant k$, then there exists some $j \in I'_n$ such that $|U_j^X| \geqslant k+1$.*

Proof. Without loss of generality, we can assume that H is connected. For sake of simplicity, we write U_j for U_j^X. Let W_i be the set of positions which symbol i appears in vertices in X excluding the first position, that is, $W_i = \{j \in I'_n : \ i \in U_j\}$.

We use the second hierarchical structure of S_n stated in Lemma 2.3 to prove the lemma by induction on $n\,(\geqslant k+1)$.

If $n = k+1$, then $\delta(H) \geqslant k = n-1$, and so $H = S_n$. Since $|U_1| = |U_2| = \cdots = |U_n| = n = k+1$, the conclusion is hold for $n = k+1$. We assume the conclusion is true for $n-1$ with $n \geqslant k+2$.

Let $x = p_1p_2\cdots p_n$ be a vertex in H. Then $x \in V(S_n^{1:p_1})$. By the second hierarchical structure, all the neighbors of x are in different $S_n^{j:p_1}$ for each $j \in I'_n$. Since $\delta(H) \geqslant k$, p_1 appears in at least k different positions of vertices in H excluding the first position. It follows that

$$|W_{p_1}| \geqslant k \quad \text{for any } x = p_1p_2\cdots p_n \in X \tag{3.1}$$

If $|U_1| = n$, then each symbol of I_n appears in the first position of vertices in H. By (3.1), we have

$$|W_i| \geqslant k \quad \text{for each } i \in I_n. \tag{3.2}$$

Now we construct an $n \times (n-1)$ matrix $C = (c_{ij})_{n\times(n-1)}$, where

$$c_{ij} = \begin{cases} 1 & j+1 \in W_i \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} |U_j| &= \sum_{i=1}^{n} c_{ij} \quad \text{for each } j \in I'_n \text{ and} \\ |W_i| &= \sum_{j=2}^{n} c_{ij} \quad \text{for each } i \in I_n. \end{aligned}$$

It follows that

$$\sum_{j=2}^{n} |U_j| = \sum_{j=2}^{n}\sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n}\sum_{j=2}^{n} c_{ij} = \sum_{i=1}^{n} |W_i|. \tag{3.3}$$

Combining (3.3) with (3.2), we have

$$\sum_{j=2}^{n} |U_j| = \sum_{i=1}^{n} |W_i| \geqslant nk. \tag{3.4}$$

If $|U_j| \leqslant k$ for each $j \in I'_n$, then $(n-1)k \geqslant nk$ by (3.4), a contradiction. Thus, there exists some $j \in I'_n$ such that $|U_j| \geqslant k+1$.

If $|U_1| < n$, then there exists at least one symbol in I_n that does not appear in the first position of any vertex in H. Without loss of generality, assume $1 \notin U_1$. Then $S_n^{1:1}$ does not contain vertices of H. By the second hierarchical structure, H must be contained in the unique $S_n^{j_0:1}$ for some $j_0 \in I'_n$ since H is connected. Because $S_n^{j_0:1}$ is isomorphic to S_{n-1}, and $H \subseteq S_n^{j_0:1}$, by the induction hypothesis, there exist some $j \in I'_n$ such that $|U_j| \geqslant k+1$.

By the induction principle, the lemma follows. ∎

Lemma 3.2 *For any integer k with $0 \leqslant k \leqslant n-2$, $\kappa_s^{(k)}(S_n) \leqslant (k+1)!(n-k-1)$ and $\lambda_s^{(k)}(S_n) \leqslant (k+1)!(n-k-1)$.*

Proof. Let

$$X = \{\, p_1\cdots p_{k+1}12\cdots(n-k-1):\ p_i \in I_n \setminus I_{n-k-1} \text{ for each } i \in I_{k+1}\}.$$

Then, the subgraph H of S_n induced by X is isomorphic to S_{k+1}. Let T be the set of neighbors of X in $S_n - X$ and F the set of edges between X and T. By the definition of S_n,

$$\begin{aligned} T = \ & \{ip_2\cdots p_{k+1}12\cdots(i-1)p_1(i+1)\cdots(n-k-1): \\ & \qquad i \in I_{n-k-1},\ p_j \in I_n \setminus I_{n-k-1} \text{ for } j \in I_{k+1}\}. \end{aligned}$$

For a vertex of X, since it has k neighbors in X, it has exactly $(n-k-1)$ neighbors in T. In addition, it is easy to see that every vertex of T has exactly one neighbor in X. It follows that

$$|T| = |F| = (k+1)!(n-k-1).$$

Since every vertex v in $S_n - X$ has at most one neighbor in X, v has at least $((n-1)-1 \geqslant) k$ neighbors in $S_n - X$, which implies that F is a k-edge-cut of S_n. It follows that

$$\lambda_s^{(k)}(S_n) \leqslant |F| = (k+1)!(n-k-1).$$

We now show that T is a k-vertex-cut of S_n. To this end, we only need to show that every vertex in $S_n - (X \cup T)$ has at least k neighbors within.

Let u be arbitrary vertex of $S_n - (X \cup T)$. We need to show that at most one of neighbors of u is in T. Suppose to the contrary that u has two distinct neighbors v and w in T. Then the first digits of v and w are different. Without loss of generality, assume $v = 1p_2 \ldots p_{k+1}p_1 23 \cdots (n-k-1)$ and $w = 2p'_2 \ldots p'_{k+1}1p'_2 3 \cdots (n-k-1)$. Since u is adjacent to v, then u and v have exactly one digit difference excluding the first one. So are u and w. Therefore, w and v have exactly two digits difference excluding the first one. But w and v have yet two digits(the $(k+2)$-th and the $(k+3)$-th) difference, then $p_2 \ldots p_{k+1} = p'_2 \ldots p'_{k+1}$, therefore $v = w$, a contradiction.

Since u has at most one neighbor in T, u has at least $((n-1)-1 \geqslant) k$ neighbors in $S_n - (X \cup T)$, which implies that T is a k-vertex-cut of S_n. It follows that

$$\kappa_s^{(k)}(S_n) \leqslant |T| = (k+1)!(n-k-1).$$

The lemma follows. ∎

Theorem 3.3 $\kappa_s^{(k)}(S_n) = \lambda_s^{(k)}(S_n) = (k+1)!(n-k-1)$ *for any* k *with* $0 \leqslant k \leqslant n-2$.

Proof. By Lemma 3.2, we only need to show that, for any k with $0 \leqslant k \leqslant n-2$,

$$\lambda_s^{(k)}(S_n) \geqslant (k+1)!(n-k-1) \text{ and } \kappa_s^{(k)}(S_n) \geqslant (k+1)!(n-k-1). \tag{3.5}$$

We prove (3.5) by induction on k. If $k = 0$, then $\lambda_s^{(0)}(S_n) = \lambda(S_n) = n-1$ and $\kappa_s^{(0)}(S_n) = \kappa(S_n) = n-1$ by Lemma 2.1, and so (3.5) is true for $k = 0$. Assume (3.5) holds for $k-1$ with $k \geqslant 1$, that is, for any k with $1 \leqslant k \leqslant n-2$,

$$\kappa_s^{(k-1)}(S_{n-1}) \geqslant k!(n-k-1) \ \text{ and } \ \lambda_s^{(k-1)}(S_{n-1}) \geqslant k!(n-k-1). \tag{3.6}$$

Let T be a minimum k-vertex-cut (or k-edge-cut) of S_n. We show that

$$|T| \geqslant (k+1)!(n-k-1) \ \text{ for } 1 \leqslant k \leqslant n-2. \tag{3.7}$$

Let X be the vertex-set of a connected component H of $S_n - T$, and

$$Y = \begin{cases} V(S_n - (X \cup T)) & \text{if } T \text{ is a vertex-cut;} \\ V(S_n - X) & \text{if } T \text{ is an edge-cut.} \end{cases}$$

Then $\delta(H) \geqslant k$, and so there exists some $j \in I'_n$ such that $|U_j^X| \geqslant k+1$ by Lemma 3.1. We choose $j_0 \in \{j \in I'_n : \ |U_j^X| \geqslant k+1\}$ such that $|U_{j_0}^X \cap U_{j_0}^Y|$ and $|U_{j_0}^Y|$ are as large as

possible. Without loss of generality, assume $j_0 = n$. In the following proof, we use the first hierarchical structure stated in Lemma 2.2. Let, for $i \in I_n$,

$$X_i = X \cap V(S_{n-1}^{n:i}), \quad Y_i = Y \cap V(S_n^{n:i}),$$
$$T_i = \begin{cases} T \cap V(S_n^{n:i}) & \text{if } T \text{ is a vertex-cut;} \\ T \cap E(S_n^{n:i}) & \text{if } T \text{ is an edge-cut,} \end{cases}$$

and let

$$J_X = \{i \in I_n : \ X_i \neq \emptyset\}, \quad J_Y = \{i \in I_n : \ Y_i \neq \emptyset\}, \ J_0 = J_X \cap J_Y.$$

Clearly, $|J_X| = |U_n^X|, |J_Y| = |U_n^Y|$ and $|J_0| = |U_n^X \cap U_n^Y|$.

If $i \in J_0$, T_i is a vertex-cut (or an edge-cut) of $S_n^{n:i}$. For any vertex x in $S_n^{n:i} - T_i$, since x has degree at least k in $S_n - T$ and has exactly one neighbor outsider $S_n^{n:i}$, x has degree at least $k-1$ in $S_n^{n:i} - T_i$. Therefore, T_i is a $(k-1)$-vertex-cut (or a $(k-1)$-edge-cut) of $S_n^{n:i}$ for any $i \in J_0$. By the induction hypothesis (3.6), we have

$$|T_i| \geqslant k!(n-k-1) \ \text{ for each } i \in J_0. \tag{3.8}$$

If $|J_0| \geqslant k+1$, by (3.8) we have

$$|T| = \sum_{i=1}^{n} |T_i| \geqslant \sum_{i \in J_0} |T_i| \geqslant (k+1)k!(n-k-1) = (k+1)!(n-k-1),$$

and so (3.7) follows.

Now assume $|J_0| \leqslant k$. Then $J_X \setminus J_0 \neq \emptyset$. We consider two cases, $J_Y \setminus J_0 \neq \emptyset$ and $J_Y \setminus J_0 = \emptyset$, respectively.

Case 1. $J_Y \setminus J_0 \neq \emptyset$,

Assume $j_1 \in J_X \setminus J_0, j_2 \in J_Y \setminus J_0$. Then there are $(n-2)!$ independent edges between $S_n^{n:j_1}$ and $S_n^{n:j_2}$. Since each vertex in $S_n^{n:j_1}$ has unique external neighbor, thus $\bigcup_{j_1 \in J_X \setminus J_0} S_n^{n:j_1}$ and $\bigcup_{j_2 \in J_Y \setminus J_0} S_n^{n:j_2}$ have $|J_X \setminus J_0||J_Y \setminus J_0|(n-2)!$ independent edges between them. Note that each edge of which must have one end-vertex in T if T is a vertex-cut, and each edge of which is contained in T if T is an edge-cut. Therefore, no matter T is a vertex-cut or an edge-cut, we have

$$\sum_{i \in (J_X \cup J_Y) \setminus J_0} |T_i| \geqslant |J_X \setminus J_0||J_Y \setminus J_0|(n-2)!. \tag{3.9}$$

Let

$$a = |J_X \setminus J_0|, \ b = |J_Y \setminus J_0|, \ c = |I_n \setminus (J_X \cup J_Y)|.$$

Then $a \geqslant 1, b \geqslant 1, a+b+c = n - |J_0|$, and so

$$\begin{aligned} ab + c = & \ ab + (n - |J_0|) - (a+b) \\ = & \ (n - |J_0|) + (a-1)(b-1) - 1 \\ \geqslant & \ (n - |J_0| - 1), \end{aligned}$$

that is,

$$ab + c \geqslant (n - |J_0| - 1). \tag{3.10}$$

Note that $c = 0$ if T is an edge-cut. Thus if there exists some $i \in I_n \setminus (J_X \cup J_Y)$, then T is a vertex-cut and $T_i = S_n^{n:i}$, and so

$$|T_i| = (n-1)! \ \text{ if } i \in I_n \setminus (J_X \cup J_Y). \tag{3.11}$$

Thus, no matter T is a vertex-cut or an edge-cut. Combining (3.8), (3.9) and (3.11) with (3.10), we have that

$$\begin{aligned}
|T| &= \sum_{i=1}^{n} |T_i| \geqslant \sum_{i \in J_0} |T_i| + \sum_{i \in (J_X \cup J_Y) \setminus J_0} |T_i| + \sum_{i \in I_n \setminus (J_X \cup J_Y)} |T_i| \\
&\geqslant |J_0| k!(n-k-1) + ab(n-2)! + c(n-1)! \\
&\geqslant |J_0| k!(n-k-1) + (ab+c)(n-2)! \\
&\geqslant |J_0| k!(n-k-1) + (n-|J_0|-1)(n-2)! \\
&\geqslant (n-1)k!(n-k-1) \\
&\geqslant (k+1)!(n-k-1),
\end{aligned}$$

and so (3.7) follows.

Case 2. $J_Y \setminus J_0 = \emptyset$,

In this case $J_Y = J_0$, then $|U_n^Y| = |J_Y| \leqslant k$. Let $\overline{X}_i = S_n^{n:i} - X_i$ for each $i \in I_n \setminus J_0$. Note that for each $i \in I_n \setminus J_0$, $\overline{X}_i = T_i$ if T is a vertex-cut, and $\overline{X}_i = \emptyset$ if T is an edge-cut.

We first show there is no $i \in I_n \setminus J_0$ such that $|\overline{X}_i| < (n-2)!$. Suppose to the contrary that there exists some $i \in I_n \setminus J_0$ such that $|\overline{X}_i| < (n-2)!$.

We show $|U_j^{X_i}| \geqslant n-1$ for any $j \in I'_{n-1}$. On the contrary, there exists some $j \in I'_{n-1}$ such that $|U_j^{X_i}| \leqslant n-2$. Notice that the rightmost digit of every vertex in X_i is i. There is at least one symbol $i_1 \in I_n \setminus \{i\}$ that does not appear in the j-th position of any vertex in X_i. Thus, the vertices with symbol i_1 in the j-th position and symbol i in the n-th position are not contained in X_i, which means that $\overline{X}_i$ contains at least $(n-2)!$ vertices, that is, $|\overline{X}_i| \geqslant (n-2)!$, a contradiction. Thus, $|U_j^{X_i}| \geqslant n-1$, and so $|U_j^X| \geqslant n-1$ for any $j \in I'_{n-1}$.

Since $\delta(Y) \geqslant k$ and $|U_n^Y| \leqslant k$, by Lemma 3.1 there exists some $j_1 \in I'_{n-1}$ such that $|U_{j_1}^Y| \geqslant k+1$. Then $|U_{j_1}^X| \geqslant n-1$ and $|U_{j_1}^Y| \geqslant k+1$, and so $|U_{j_1}^X \cap U_{j_1}^Y| \geqslant k$ and $|U_{j_1}^Y| \geqslant k+1$. Note that $|U_n^X \cap U_n^Y| = |J_0| \leqslant k$ and $|U_n^Y| = |J_Y| = |J_0| \leqslant k$. This contradicts to the choice of $j_0 (= n)$.

Thus, there is no $i \in I_n \setminus J_0$ such that $|\overline{X}_i| < (n-2)!$, and so there is no $i \in I_n \setminus J_0$ such that $|\overline{X}_i| = 0$. If T is an edge-cut, then $\overline{X}_i = \emptyset$, a contradiction. Therefore, T is a vertex-cut, and so $\overline{X}_i = T_i$. It follows that

$$|T_i| = |\overline{X}_i| \geqslant (n-2)! \ \text{ for each } i \in I_n \setminus J_0. \tag{3.12}$$

Combining (3.12) with (3.8), we have

$$\begin{aligned}
|T| &= \sum_{i=1}^{n} |T_i| = \sum_{i \in J_0} |T_i| + \sum_{i \in I_n \setminus J_0} |T_i| \\
&\geqslant |J_0| k!(n-k-1) + (n-|J_0|)(n-2)! \\
&\geqslant (k+1)!(n-k-1).
\end{aligned}$$

By induction principles, (3.7) holds and so the theorem follows. ∎

Corollary 3.4 ([10], [11]) *$\kappa_s^{(2)}(S_n) = \lambda_s^{(2)}(S_n) = 6(n-3)$ for $n \geqslant 4$.*

4 Conclusions

In this paper, we consider the generalized measures of fault tolerance for networks, called the k-super connectivity $\kappa_s^{(k)}$ and the k-super edge-connectivity $\lambda_s^{(k)}$. For n-dimensional star graph S_n, which is an attractive alternative network to hypercubes, we prove that $\kappa_s^{(k)}(S_n) = \lambda_s^{(k)}(S_n) = (k+1)!(n-k-1)$ for $0 \leqslant k \leqslant n-2$, which gives an affirmative answer to the conjecture proposed by Wan and Zhang [10]. The results show that at least $(k+1)!(n-k-1)$ vertices or edges have to be removed from S_n to make it disconnected and no vertices of degree less than k. Thus these results can provide more accurate measurements for fault tolerance of the system when n-dimensional star graphs is used to model the topological structure of a large-scale parallel processing system.

References

[1] S. B. Akers, B. Krishnamurthy, A group theoretic model for symmetric interconnection networks. IEEE Transactions on Computers, 38 (4) (1989), 555-566.

[2] E. Cheng, M. J. Lipman, Increasing the connectivity of the star graphs. Networks, 40 (3) (2002), 165-169.

[3] K. Day, A. Triphthi, A comparative study of topological properties of hypercubes and star graphs. IEEE Transactions on Parallel and Distributed Systems, 5 (1) (1994), 31-38.

[4] A. H. Esfahanian, Generalized measures of fault tolerance with application to n-cube networks. IEEE Transactions on Computers, 38 (11) (1989), 1586-1591.

[5] S.-C. Hu, C.-B. Yang, Fault tolerance on star graphs. International Journal of Foundations of Computer Science, 8 (2)(1997), 127-142.

[6] S. Latifi, M. Hegde, M. Naraghi-Pour, Conditional connectivity measures for large multiprocessor systems. IEEE Transactions on Computers, 43 (2) (1994), 218-222.

[7] W. Shi, F. Luo, P. K. Srimani, A new hierarchical structure of star graphs and applications. In: Distributed Computing and Internet Technology, LNCS, 7154, (2012), pp. 267-268.

[8] Y. Rouskov, S. Latifi, P. K. Srimani, Conditional fault diameter of star graph networks. Journal of Parallel and Distributed Computing, 33 (1) (1996), 91-97.

[9] D. Walker, S. Latifi, Improving bounds on link failure tolerance of the star graph. Information Sciences, 180 (13) (2010), 2571-2575.

[10] M. Wan, Z. Zhang, A kind of conditional vertex connectivity of star graphs. Applied Mathematics Letters, 22 (2009), 264-267.

[11] W.-H. Yang, H.-Z Li, J.-X Meng, Conditional connectivity of Cayley graphs generated by transposition trees. Information Processing Letters, 110 (23) (2010), 1027-1030.